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ABSTRACT

THIS IS ONE CF A SERIES OF UNITS INTENDED FOR BOTH PRESERVICE AND INSERVICE ELEMENTARY SCHOOL TEACHERS TO SATISFY A NEED FOR MATERIALS ON "NEW MATHEMATICS" PROGRAMS WHICH (1) ARE READABLE ON A SELF BASIS OR WITH MINIMAL INSTRUCTION, (2) SHOW THE PEDAGOGICAL OBJECTIVES AND USES OF SUCH MATHEMATICAL STRUCTURAL IDEAS AS THE FIELD AXIOMS, SETS, AND LOGIC, AND (3) RELATE MATHEMATICS TO THE "REAL WORLD," ITS APPLICATIONS, AND CTHER AREAS OF CURRICULUM. THIS UNIT EXPLORES BASIC COUNTING AND CRDER CONCEPTS PRIMARILY FROM AN INFORMAL, BUT ESSENTIALLY LOGICAL POINT OF VIEW. PART 1 DESCRIBES THREE EXPERIMENTS PERFORMED BY PIAGET ET AL., IN ORDER TO UNDERSTAND BETTER THE PROCESS A CHILD GOES THROUGH IN ATTAINING A WORKING FAMILIARITY WITH NUMBER CONCEPTS - (1) AN EXPERIMENT INVOLVING THE BASIC CONCEPTION OF "MORE THAN" WITHOUT NUMBER IDEAS, (2) AN EXPERIMENT INVOLVING ONE-ONE CORRESPONDENCE, AND (3) AN EXPERIMENT INVOLVING CONCEPTS ON SETS AND CARDINAL NUMBERS. PART 3 SHOWS THE INTER-RELATIONSHIP BETWEEN COUNTING AND CRDER. (BP)

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BASIC NUMBER AND ORDER IDEAS

USOE Project - University of Michigan

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OFFICE OF EDUCATION

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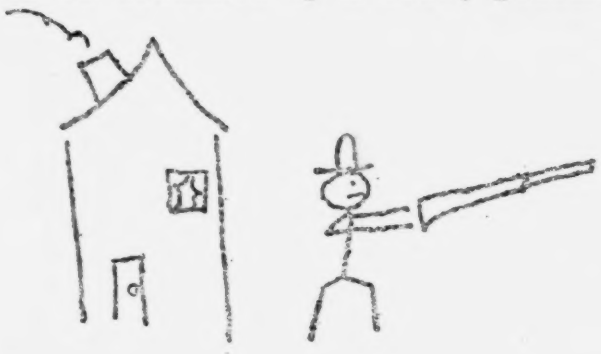
Unit 1. Introduction

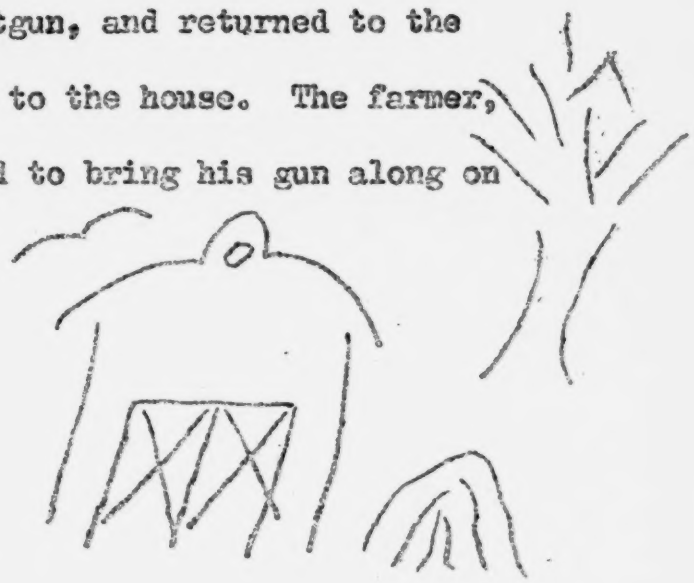
NATURE OF THE PROBLEM

The concept of the "counting numbers", 1,2,3,..., which are known to mathematicians as "natural numbers" or "positive integers", is deeply ingrained in most of us. So deeply, in fact, that it is sometimes difficult for us to understand how a child learns this concept and to appreciate the trouble that the human race has encountered in utilizing number ideas. Consider, for example, a kindergarten teacher whose class is given dixie cups by a donor uncertain of the size of the class. The teacher quickly determines which is more: the number of dixie cups, or the number of kindergarteners. In order to determine this, she counts the dixie cups and visualizes a one-one correspondence between the set of kindergarteners and (hopefully) a subset of the dixie cups.

In order to see more fully the differences among the three notions underlined above, and the relations among them, consider these situations.

1. Levi Conant in The Number Concept tells of a farmer who was determined to be rid of a certain pesky crow. The crow, however, flew away whenever the farmer left the house with his shotgun, and returned to the corn crib only when the farmer returned the gun to the house. The farmer, not to be outfoxed by a crow, persuaded a friend to bring his gun along on





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his next visit. When the friend arrived, he and the farmer started for the barn, guns in hand. The crow flew off, to watch the proceedings from the safety of a distant tree. The farmer returned his gun to the house while his friend waited in the barn for the crow. But the crow remained at a distance, until the farmer's friend returned to the house in disgust.

The farmer's next move was to call the members of the NFO (National Farm Organization), who were to meet at his house the following weekend, and invite them to bring their shotguns. When they arrived, three farmers, with shotguns, headed for the barn, where one remained while the other two returned. From the safety of the tree, the crow commented "caw". The farmers tried again, sending four of their members to the barn. Still the crow wasn't fooled. Finally, when five armed farmers entered the barn and four of them returned to the house, the crow flew up from his tree and down toward the corn-crib and his doom.

In this story, the crow was presented with a simple more than problem: to determine whether the number of farmers entering the barn was more than the farmers returning to the house. The problem was clear, the technique of solution was lacking.

2. A neighboring tribe threatens to attack a primitive village. The chief of the village decides to stand and fight if his warriors are more than the attacking warriors, and to retreat otherwise. He sends a scout to determine the size of the enemy force. The scout returns and reports: "many". The chief perceives this as an inadequate reply, but is at a loss as to what to do about it, since the tribe's language is undeveloped and has only three number words: one, two, and many. The village wiseman is summoned to attack this technical problem. He sends the scout out with a pouch of pebbles and an empty pouch. For each enemy the scout transfers

one pebble to the empty pouch. He then returns and presents the pouch with the transferred pebbles to the chief.

Again the problem was clear, but the technique for its solution was awkward. Counting would have saved the crow and helped the chief.

We can guess, from speculating about situations such as the ones above, that number concepts were developed in close relation to the real situations in which they were needed. On the other hand, the child of today normally learns to count before he has much number understanding. In fact, he often memorizes a sequence of number words before he has any conception of their significance. To a child of two or three years, abbreviating "1,2,3,4,5" to "2,4,5" is no different than abbreviating "My dolly is sleeping now" to "dolly sleep now". Further, intelligent use of counting involves much more than memorizing a sequence of words in the proper order. For example, as a child is taught to enumerate sets of objects with the number sequence, he may at first think that he is being taught names for the objects, as is the case when his mother points to a chair and says, "chair". It is common for young children, in early imitative experimenting with the process of counting objects, to say the number words over a set of objects with no apparent thought of matching one number with each object.

SOME PSYCHOLOGICAL EVIDENCE

Psychologists have studied the understandings and partial understandings of number concepts gained by children at various stages in their development. Without attempting a full analysis of the psychological theories involved, we can, by examining some of the work of these psychologists, gain insight into the kind of thought that children apply to number situations. Specifically, let us look at a few of the experiments performed by Piaget[1]

1. Jean Piaget, The Child's Conception of Number. (New York, The Humanities Press Inc., 1952) p. G-64

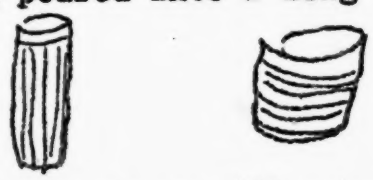
and his associates in order to understand better the process a child goes through in attaining a working familiarity with number concepts. Keeping these experiments in mind while reading Unit II should also allow a comparison of the development of number ideas by children with the same ideas developed from a more mathematical and logical point of view.

<u>An Experiment involving the Basic conception of "more than" without number ideas:</u>

Clairette, a four-year-old girl, is given a glass ¾ full of orangeade and her companion is given a similar glass ¾ full of lemonade.

Clairette agrees that both glasses have the same amount "to drink". In sight of the children, all the orangeade is poured into two smaller glasses, and the experimenter asks which is more, the orangeade or the lemonade.

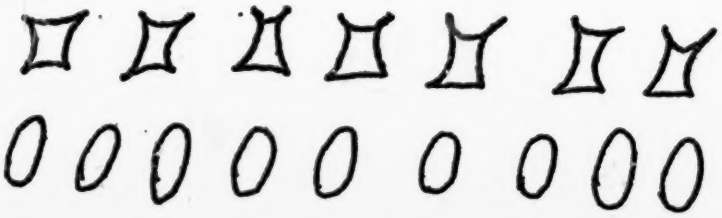
Clairette now asserts that there is more orangeade. She maintains this in spite of further questioning. The experiment continues with the orangeade being poured into a long thin tube.



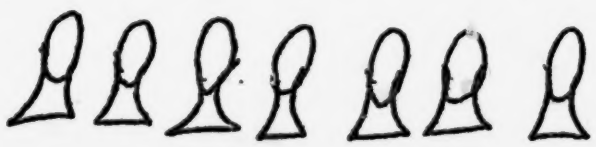
In this case the four-year-old asserts that there is more orangeade than lemonade. When asked why, she says, "We've poured it into that glass." Finally, the orangeade is poured into two glasses, and the lemonade is poured into two similar glasses. At this Clairette says, with conviction, "They are the same."

An Experiment involving One-One Correspondence:

Seven egg-cups are placed on a table in front of a boy of four years, nine months. He is directed to "take just enough eggs for the egg-cups, not more and not less, one egg for each cup". The child begins by putting next to the egg-cups a row of eggs of the same length, but containing too many eggs.



The eggs are then put in the cups and the extra ones removed.



The seven eggs are removed from the cups, and put into a group.

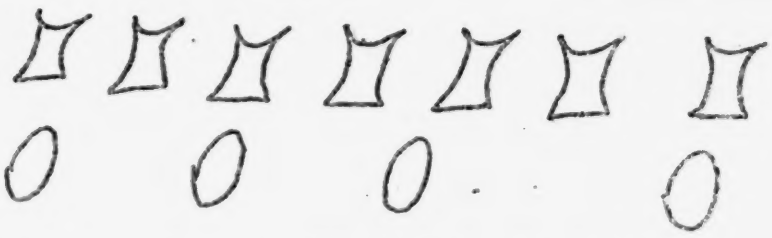
The experimenter asks, "Is there the same number of eggs and egg-cups?"

"No, there are a lot of egg-cups and less eggs."

"Are there enough eggs for the egg-cups?"

"No."

All of the eggs are removed and four are returned in a row of the same length as the row of cups.



"Are there enough eggs for these egg cups?"

"Yes."

"Put them in yourself and see."

The boy tries and seems very much surprised that there are not enough.

On another trial, with only three eggs for seven cups, the correct response, "There'll be some empty egg-cups" is given.

<u>An Experiment involving both counting and one-one correspondence:</u>

The experimenter and a 5½ year-old boy play a game in which the experimenter sells sweets for 1¢ per candy. The child is given eight pennies to buy sweets. He counts the pennies and exchanges them in a one for one manner for sweets.

The experimenter asks, "How many sweets have I given you?"

The boy counts, "Eight."

The experimenter hides the pennies under his hand.

He asks, "How many pennies have you given me?"

The boy answers, "Ten."

Similar experiments indicate that before a child will answer correctly questions of this type, requiring use of one-one correspondence combined with counting to establish the number of elements in a set, he progresses through stages of answering correctly for certain numbers, but not for larger ones. For example, a child may respond correctly for sets of three pennies and sweets, but fail to do so for seven.

Although our main purpose, that of gaining some insight into a child's development of understanding of number concepts, may be adequately served by considering these experiments in isolation, let us briefly pursue a portion of the psychological theory explaining the behavior described. This theory centers around the notion of "conservation". Two types of conservation involved in the experiments described above are conservation of <u>amount</u> and conservation of <u>number</u>.

In situations such as the lemonade experiment, a child is said to have achieved conservation when he realizes that the amount of liquid does not depend on the number or shape of the containers which hold it. In performing his experiments, Piaget observed that a child passes through stages of "partial conservation", characterized by conflict between direct perception and reason. For example, in the lemonade experiment, a child may admit that the amount of liquid in a glass is conserved when it is poured into two similarily shaped glasses, but be overcome by his perception if the same liquid is poured into four tall, thin glasses, which makes it look to him as though the amount of liquid has increased.

The idea of amount is a mathematical-physical concept which may produce conflict when a child is learning about number. In some situations, the "more than" relation holds between two amounts or physical quantities such as volumes or distances, but fails to hold between a related pair of numbers. For example, you may get further with two giant steps than with three baby steps.

A child has developed conservation of number for a particular set of objects when he knows that the number of elements in the set is unaltered by manipulation of the elements of the set (assuming, of course, that the elements themselves are left intact.) In the egg-cup experiment, a child would demonstrate this conservation by realizing that the number of eggs in a set is the same whether the eggs are in a pile or stretched out in a long line. Partial conservation would be evidenced by a child's conserving the number of a set of four eggs, but not of a set of seven; or by conserving the number of eggs in a pile when the eggs are arranged in a short line, but not when the eggs are used to make a long line.

This brief explanation of some of Piaget's work does not give an

adequate description of his theory. It suggests, however, that even the most fundamental number concepts are not grasped at once by children, but attained slowly, through experience.

In this section, we have speculated about the historical development of number concepts, and we have looked at one theory of the development of number concepts in individuals. In the next section, the mathematical ideas inherent in the natural numbers will be more carefully investigated.

EXERCISES AND DISCUSSION QUESTIONS

1. In the crow story, a sufficiently clever bird might have used some device to help him solve his problem. How might he have used
   a. his toes?
   b. some acorns?
   c. his beak to mark the tree?
   d. some other device?

2. In the primitive king story, the king decides to divide his army into three equal parts, one for defense, one to attack the enemy camp, and one to protect his sacred person.
   a. How could he use the pebbles to perform this division?
   b. Concoct a situation for the king requiring addition and another requiring subtraction, and describe a solution using pebbles.
   c. Conjure up a division situation for the king, in which the given information includes the number in each group, but not the number of groups.

3. Conservation questions:
   a. What kinds of partial conservation would you expect children to develop in a situation such as the candy buying experiment described in this unit?

b. Suggest another conservation of number experiment, and speculate about its outcome.

c. From your experience, describe some difficulties you have seen young children have in understanding basic number ideas. Are the difficulties explanable in terms of conservation?

d. By what age do you think most (90%) of children conserve quantity? The number of elements in a set?

Unit 2. Sets and Cardinal Number

INTRODUCTION

The modern mathematical attempts to analyze the natural numbers are attempts to break down the number concept into component fundamental ideas. The relation between the fundamental concepts isolated by mathematicians and the fundamental concepts which a child must attain in the development of useable number ideas is of considerable interest. In order to discuss this relation intelligently, some understanding of the set theory discussed in this unit will be needed.

SET AND ELEMENT

The most fundamental pre-number concept in mathematics is that of a <u>set</u>. A set is a collection of objects which are called the <u>elements</u> of the set. As usual in mathematics, the fundamental terms are repeated often, so that some symbolic abbreviation is helpful. Thus, a set is often described by listing its elements between braces, $\{\ \}$. For example, $\{a, e, i, o, u\}$ is the set whose elements are the letters in our alphabet which are always vowels. We can give this set a name by letting $A = \{a, e, i, o, u\}$. By introducing the symbol $\in$ for the words "is an

element of" we can simply say "a∈A" rather than the longer "a is an element of the set whose elements are a, e, i, o, u".

The crucial property of a set is that its elements be unambiguously identified. Thus, whenever a specific set is being considered, it should be made clear what the elements of the set are. In particular, since y is SOMETIMES but not always a vowel, to speak of the "set of vowels" is ambiguous, that is, no set is determined by the words in quotation marks.

Possibly some intuition into basic set notation can be gained by considering briefly the motivating philosophy for the creation of set theory. The idea of set formation was conceived as a formalization of one of the most fundamental intellectual processes: the process of gathering together a collection of things into a single category. Basic set notation is meant to convey this fundamental notion, and <u>nothing more</u>.

In keeping with this outlook, $\{a,b,c,\}$ and $\{b,c,a,\}$ are the same set, i.e. $\{a,b,c,\} = \{b,a,c\}$. We are forced into two symbols for the same set by the unavoidability of writing symbols for the elements in some order.

Another ambiguity that may arise from the abstraction of symbolization is the ambiguity of repeating the same symbol. Is $\{a,b\}$ the same set as $\{a,a,b\}$? If a child has a red pencil and a blue pencil, the set whose elements are the pencils is directly perceived without symbols, and the symbolic difficulty does not arise. The accepted attitude toward sets like $\{a,a,b\}$ is that there is only one (small) "a" in the alphabet, and hence $\{a,a,b\} = \{a,b\}$. $\{a,a,b\}$ is not incorrect, but awkward, and is generally avoided, just as you would avoid entering a student's name twice in your grade-book.

There is another situation which arises because of our use of symbols to denote the elements of a set, and which needs to be taken into account in teaching children about sets. There are instances in which maintaining a careful one-one correspondence between symbols and objects becomes awkward. For example, if we wish to picture the set of John's marbles, we might use the symbols $\{\circledcirc,\circledcirc,\circledcirc,\circledcirc,\circledcirc\}$ rather than $\{o,o,o,o,o\}$, to indicate that there are actually five different marbles not merely one marble pictured five times. On the other hand, sometimes the desire to use precise symbolism should be overridden. Answering the question "What are the funny marks on the marbles?" might not be a good starting place for a lesson about sets or addition, and children will normally assume that the marbles are different even though the typed symbols look alike. The point is that if a set is to be discussed, you, as learner or as teacher, should make certain that it is clear precisely what the elements of the set are.

The symbol "=" is used in a somewhat special sense in set theory. While in some contexts it might be proper to say that

a dozen marbles = a dozen pencils,

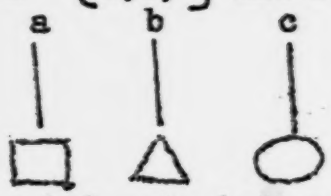
in set theory, "=" is used only to mean that the two sets mentioned are actually the same set. Thus $\{a,a,b\} = \{b,a\}$ means that $\{a,a,b\}$ and $\{b,a\}$ are actually the same set.

Set theory allows a set that has no elements, which is called $\emptyset$, the empty set. Using this symbol, we can express the fact that there are no elements common to $\{a,b\}$ and $\{c,d\}$ by saying that $\emptyset$ is the set of elements which are in both $\{a,b\}$ and $\{c,d\}$. This special and very rudimentary set, $\emptyset$, has an important role in the mathematical development of number.

Let us now use the convenience of the symbols of set theory in examining fundamental number ideas.

CARDINAL NUMBER

The approach to numbers used by the primitive king in the first unit is studied in mathematics through one to one correspondence. Two sets can be put into one to one correspondence if their elements can be matched. For example, the diagram below represents a one to one correspondence between the sets A = {a,b,c} and B = {□,△,○}



We will use the symbol ~ to indicate that two sets can be put into one to one correspondence. In the example above, A~B is read A is <u>equivalent</u> to B. In contrast, A = B means that A and B are different names for the same set. With an eye to psychological theory and its implications for the classroom, we note that A~B is also demonstrated by both of the following diagrams:

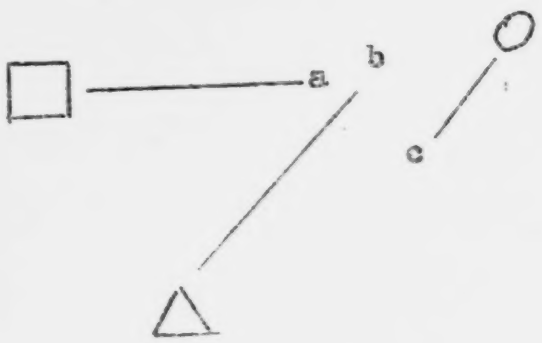


Figure 1.

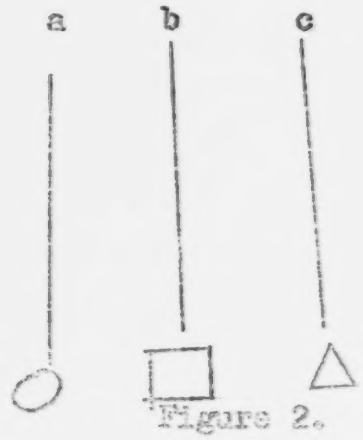


Figure 2.

This indicates that a one to one correspondence between two sets may be set up in many different ways, but each demonstrates the same mathematical fact that the two sets are equivalent.

We can also make formal psychological statements about conservation in these situations:

Conservation over pairs used is achieved when a child recognizes that the equivalence of two sets cannot be destroyed by changing the matching used. For example,

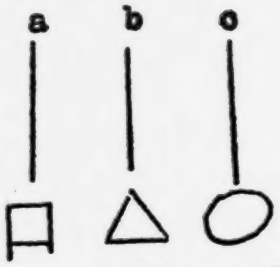


Figure 3.

and

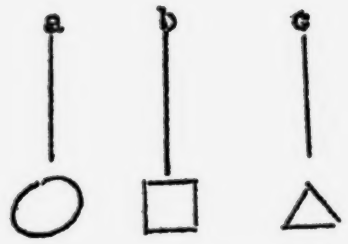


Figure 4.

both show that A~B.

Conservation over configuration is achieved when a child recognizes that the equivalence of two sets cannot be destroyed by moving the elements of the sets. For example,

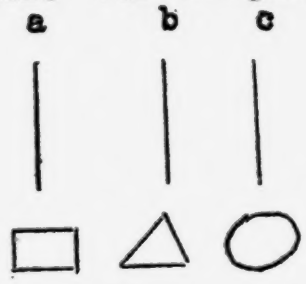


Figure 5.

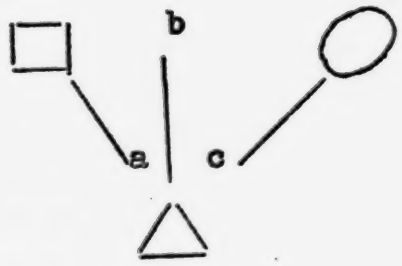


Figure 6.

Configuration conservation is a psychological and pedagogical concept, but not a concept of mathematical theory. As discussed earlier, the basic set theory notion is the "is an element of" or "belongs to" notion. $\square \in \{\square, \triangle, \bigcirc\}$ regardless of where $\square$ is, or if it is actually anywhere. Stated more properly, this says that the location of the elements of a set is not pertinent to set theory.

A mathematical approach to number commences with the definition: two sets have the same cardinal number if they are equivalent. It is worthy of note that this definition does not explicitly define cardinal number; it is not stated in the form "a cardinal number is...". What

the definition gives is a relation between pairs of sets. If two sets are specified, the definition allows us to answer the "yes or no" question, do these two sets have the same cardinal number?

Accordingly, we can describe one property of a set by saying that it has the same cardinal number as some other set. For example, the set of fingers on my left hand can be described by saying that it has the same cardinal number as the set of toes on my left foot. It has the same cardinal number as any set of pennies which is proper change for a nickel, and the same cardinal number as the set $\{a,b,c,d,e\}$. A young child shows understanding of the cardinal number concept when he holds up two fingers and says "I want this many cookies", or when he tells how many eyes he has by holding up two fingers and says "this many". The wise man in the primitive king story also understood the concept when he had the scout tell how many enemy troops there were by handing the king the pebbles and saying "this many".

*TWO MATHEMATICAL DEFINITIONS OF NUMBER BASED ON EQUIVALENCE[1]

As developed above, cardinal number is defined by the statement,

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1. A few of the sections which follow are marked with a *. While understanding these sections requires no added background, they have two objectives beyond those of the unstarred sections. First, they give historical background concerning the mathematical development of some topics which relate directly to the number and order ideas of these units. Second, they consider number and order ideas in a more philosophical and mathematical sense than is necessary for a first consideration or for classroom use. (This is not to imply that they would not be useful.) More specifically, the first three starred sections look at attempts to answer the innocent sounding, but philosophically and mathematically troubling, question: what is a number?

As stated above, the starred sections require no added background. However, they are independent of the other sections, and constitute digressions from the main flow of ideas, so that the reader may wish to omit them on first reading.

"Two sets have the same cardinal number if they are equivalent." Although for most purposes this definition is adequate, in a logical sense it would be more satisfying to have a definition that began "A cardinal number is...". Such a definition would permit a direct answer to the question, what is a cardinal number?, rather than the somewhat indirect answer that a set has a certain cardinal number if it is equivalent to a set which is known to have that cardinal number.

One way to make such a definition is to define cardinal number as a property abstracted from equivalent sets. This viewpoint was taken by Georg Cantor in 1895:

> We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate [set] M when we make abstraction of the nature of its various elements m and of the order in which they are given.
>
> We denote the result of this double act of abstraction, the cardinal number or power of M, by $\overline{\overline{M}}$.[1] ...
>
> Of fundamental importance is the theorem that two aggregates of M and N have the same cardinal number if, and only if, they are equivalent: thus,
>
> from $M \sim N$ we get $\overline{\overline{M}} = \overline{\overline{N}}$, and
>
> from $\overline{\overline{M}} = \overline{\overline{N}}$ we get $M \sim N$.[2] ...

To contrast this with what follows, note that to Cantor, a set and a number are dissimilar in nature. To him a set is a collection of definite and separate objects,[3] while a number is an abstracted property. Cantor

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1. Cantor, Georg, Contributions to the Founding of the Theory of Transfinite Numbers. Translator: Philip E. B. Jourdain. New York: Dover Publications, Inc., 1952, p. 86.
2. Ibid. p. 87.
3. Ibid. p. 85. "By an 'aggregate' (Menge) we are to understand any collection into a whole (Zusammenfassung zu einen Ganzen) M of definite and separate objects m of our intuition or our thought."

asks, "Is not an aggregate an object outside us, whereas its cardinal number is an abstract picture of it in our mind?"[1]

In 1903, Bertrand Russell, building on earlier work of Gottlob Frege, avoided introducing the notion of abstract property in the definition of number by defining a number to be a set: "The number of a class [set] is the class of all those classes that are [equivalent] to it."[2] Thus for Cantor, 2 is a property which can be abstracted from the set {a,b}, while for Frege and Russell, 2 is the set of all sets equivalent to {a,b},

$$2 = \{\{a,b\}, \{\Delta,\odot\}, \{3,4\} \ldots\} .$$

Russell justified this approach:

> We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples: it is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive.[3]

STANDARD SETS

Let us consider further the abstraction involved in answering a question with the word "five" rather than displaying the fingers on one hand and saying "this many". When we tell a person who knows about using cardinal numbers, that set has five elements, he knows that the set we are trying to describe has the same cardinal number as the set of fingers on any well-formed hand, or any set of pennies which is proper change for a

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1. Ibid. p. 80.
2. Russell, Bertrand, Introduction to Mathematical Philosophy, London: George Allen & Unwin, LTD., 1948, p. 18.
3. Ibid.

nickel, or the set $\{v,w,x,y,z\}$. For the sake of definiteness, we may choose one such set as a standard for 5. Then, when we say that a set has 5 elements, we will mean that the set in question has the same cardinal number as the standard set we have specified for 5.

The choice of a standard set for 5 is best determined from the experience of the user. Since we have made extensive use of sets of letters, let us tentatively agree to say that any set that has the same cardinal number as $\{d,b,c,a,e\}$ has the number 5, that is to say, let us agree to choose the set $\{c,b,a,d,e\}$ as a standard for the number 5. Let us further agree to the following standards:

Any set equivalent to	has the cardinal number
$\{b,a\}$	2
$\{a\}$	1
$\emptyset$	0
$\{c,a,b\}$	3
$\{a,b,d,c\}$	4,

and so forth.

Note that, from this viewpoint, no counting is involved. If we want to know the number for the set $\{\square,\triangle,\bigcirc,\}$ which we will call D, we must find the set from the list of standard sets which is equivalent to D and give D the number for that particular standard set. Since $D \sim \{d,c,b,a\}$, the number for D is 4. This use of equivalent sets to explain number can be compared to measuring length. To measure a pencil five inches long, we match the pencil with five of the standard inch units on a ruler; in "measuring" a particular set, we match it with a standard set.

*A UNIVERSAL STANDARD

We are considering standard sets primarily to stress the one-one correspondence basis of cardinal number. However, some insight into the mathematician's use of standard sets can be gained by analogy to the physicist's use of standards for physical units.

Our selection of alphabetic sets for standards was made for convenience, as you might choose a convenient yardstick as a standard for measuring the height of a window. But scientists have recognized the need for universal standards. Universal standards for length have been internationally adopted:

> In 1889 the International Bureau of Weights and Measures prepared a number of platinum-iridium meter bars and determined with great exactness the length of each. One bar, No. 6, was adopted as the International Prototype, its length being considered 1 meter.

Recent developments have allowed even more precise standards, for instance in 1948 we find:

> Developments in spectroscopy make it possible to define length in terms of wave lengths of light. This has the advantage of providing a primary standard simultaneously available in many different places and not subject to distruction or alteration. One meter equal to 1,553,164.13 wave lengths of red cadmium radiation has gained wide recognition and is used by the International Astronomical Union.[1]

Later, in 1960, action was taken on an international basis:

> It was agreed at the Eleventh General (International) Conference on Weights and Measures to redefine the meter in terms of the wavelength of the orange-red radiation in vacuum of krypton 86 corresponding to the unperturbed transition between the 2p and 5d levels as follows:
>
> 1 meter = 1 650 763.73 wave lengths.[2]

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1. National Council of Teachers of Mathematics, The Metric System of Weights and Measures, Bureau of Publications, Teachers College, Columbia University, New York, (1948), pp. 297-298
2. "Units of Weight and Measure", United States Dept. of Commerce, Natl. Bureau of Standards, Miscellaneous Publications 233, Washington D.C., (1960), p. 3

As the physicist has attempted to root his science in fundamental physical phenomena, the pure mathematician has tried to root his domain in fundamental logical and mathematical concepts. As mentioned at the beginning of Unit II, one of the most basic mathematical concepts is the set concept, and further, the most rudimentary set of $\emptyset$, the empty set. Hence, sets built up using $\emptyset$ and the basic notion of set formation have been offered as universal standards for number. In 1923, von Neumann[1] suggested the following, which have gained some recognition:

$0 = \emptyset$

$1 = \{\emptyset\}$

$2 = \{\emptyset, \{\emptyset\}\}$

.
.
.

{each non-negative integer[2] is thus the set of all non-negative integers that precede it.)

As with physical measurement, the universal standards for number (like the other mathematical definitions of number) are not used in every-day situations, although, like the wave length definition of a meter, the proposed number standards are universally available. As the physicist refers to his universal standards when he wants to be extremely precise, so the pure mathematician refers to his universals in establishing fundamental mathematical laws.

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1. von Neumann, J. "Die Axiomatisierung der Mengenlehre." Math. Zeit., Vol. 27 (1928), 669-752

2. The term "non-negative integer" is used rather than "natural number" or "counting number" because 0 is included here.

EXERCISES AND DISCUSSION QUESTIONS

1. Let A = {r,s,t}. Give a set which is equivalent to A but not equal to A. How many different sets are equal to A?

Which experiments described in Unit I concern children who fail to conserve one-one correspondence over configuration? Explain.

3. {a,b} and {c,d} can be paired in two different ways, namely:

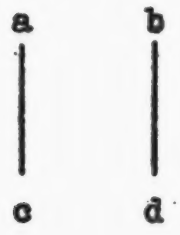

Figure 7.

and

Figure 8.

if we are considering only which pairs of elements are used. In how many different ways can {a,b,c} and {d,e,f} be paired considering only changes in pairing?

4. Using an alphabetic standard set, explain why the number of fingers on your left hand is 5.

5. Describe how the primitive king of Unit I might have set up and used standard sets for his kingdom.

6. What pairs of sets might be used by kindergartners for learning about one to one correspondence?

*7. Would the set Russell defines to be 2 serve as a standard set for 2?

Unit 3. Counting and Order

THE COUNTING PROCESS

There is an alternate way of conceiving the natural numbers which does not involve the cardinal idea of "number for a set" at its outset, as was the case in Unit II. We can investigate the rote-counting process independently of its use in enumerating elements of sets. In so doing, we are, in a sense, putting ourselves in the position of a three or four year old child who has learned the sounds "one", "two", "three", and so on, in order, but who has not learned the special use of these sounds as symbols describing the number of elements of a set. A child learns this sequence of sounds just as he learns the sequence of sounds "a", "b", "c"..., and as you can see, there is just as much in the sound "e" to suggest this many: as there is in the sound "five".

Two notions basic to the natural numbers are crucial in rote-counting:

(1) Rote-counting starts with "one".

(2) At any point in counting, there is a proper next sound. A child who rote counts "three, four, five, six" fails to comply with (1) and a child who counts "one, two, six, seven" fails to comply with (2).

*THE PEANO DEFINITION OF THE NATURAL NUMBERS

A mathematical definition of the set of natural numbers which can be seen to stem from the counting process is the Peano system (1891), which was developed by an Italian school of mathematical logicians in the latter part of the 19th century.[1] Their procedure was to define the set of

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1. Peano, G., "Sul Concetto di Numero", Rivista D I Matematica, Vol. I (1891), p. 90.

natural numbers to be a set which has certain characterizing properties, taken as axioms.

This is the same general procedure used, for example, in defining the set of children to make a trip to the fire station by requiring them to have the following properties:

(a) they are kindergartners

(b) their teacher is Mrs. Appleset

(c) they have a note of permission from home.

In a modified form, the Peano system says that the natural numbers are the elements of a set which has properties (1) and (2) of the previous section, together with

(3) once a number has occurred in the counting sequence, it won't occur again

(4) any natural number can be reached by counting far enough.

The Peano definition was one of the early axiomatic definitions of the set of natural numbers (i.e. a definition made by requiring that the object defined satisfy certain axiomatic properties, rather than a definition directly stating what the defined object is, such as the Russell-Frege definition). The Peano properties were originally stated in a technical language of symbolic logic. The properties stated above are not intended to reflect precisely the Peano system, but only to give the general flavor of the approach, in the context of our emphasis on counting. Counting was avoided in the original formulation by using the more elemental notion of <u>successor</u> (2 is the successor of 1, 7 is the successor of 6). A statement of the defining properties which is closer to Peano's original is:

(1) 1 is a number

(2) the successor of any number is a number

(3) no two numbers have the same successor

(4) 1 is not the successor of any number

(5) any property which belongs to 1, and also to the successor of every number which has the property, belongs to all natural numbers.[1]

ORDER AND COUNTING

In the preceding unit, in order to keep the basic set ideas as simple as possible, the idea of an order for the elements of a set was avoided. For example, $\{a,b,c\} = \{c,a,b\}$, and the only reason that order sneaked in at all is the necessity of writing the three letters in some order. However, in rote counting, order becomes important. Therefore, to avoid confusion, if we wish to denote that the elements of a set are to be thought of only in a certain order, we will replace the braces with parentheses. Thus, while $\{a,b,c\} = \{c,a,b\}$, $(a,b,c) \neq (c,a,b)$, i.e., (a,b,c) and (c,a,b) are different <u>ordered sets</u>.

We can now see a child who can dependably count to four as having a working acquaintance with the ordered set of sounds (one, two, three, four). Note that you have an advantage over a four or five year old in understanding the nature of this set. You have a double perception of the order in which the elements occur. First, you can hear the order as you rote count; you say "two" some time before you say "five" just as the child does. You also have a long established visual perception: you perceive that "two" is written somewhere to the left of "five", when the numbers

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1. Russell, op. cit. pp. 5-6

are written in their proper order. Equating "to the left of" with "before" is the result of much learning, as teachers of early-elementary grades well know.

We will postpone a more detailed consideration of the order idea for its own sake, and proceed directly to the connections between rote counting and cardinal number. As mentioned above, a child who can dependably count to four has a working acquaintance with the ordered set of sounds (one, two, three, four), even before he has associated any visual perception with the sounds. In fact, if the child can stop counting at will, he can say, in order, the elements of any of the ordered sets of sounds below:

(one)

(one, two)

(one, two, three)

(one, two, three, four),

(. . . . .)

and, as the child extends his rote counting, his list of ordered sets grows.

APPLICATION OF ROTE COUNTING TO CARDINAL NUMBER

As soon as the sets above are written (if not before) it becomes clear that they are the sets ordinarily used as standards in cardinal number situations. But our use of one of these sets involves more than the arbitrary selection of a set of sounds as a standard set. The saying order of the sounds in the set is crucial. A child who counts the elements of the set $\{\triangle, \bigcirc, \square\}$ in either of the saying orders indicated by the arrows has produced a one-one correspondence between (one, two, three)

Figure 9.

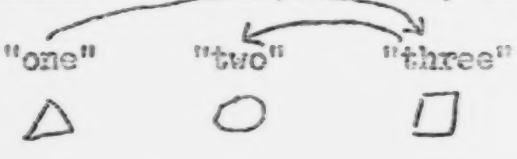

Figure 10.

and {△,□,○}, but he has not counted the elements of {○,△,□} properly. Thus the order in the standard sets is essential.

But why must this complicating notion of order be injected? We have agreed to say that a set has 3 elements if it is equivalent to {two, three, one}. Why talk of another use of three? The answer is that, by insisting on order, we can obtain the economy of using the symbol for 3 in two different ways:

(1) "Three" is the last element in the ordered set of sounds (one, two, three), and

(2) "Three" is the symbol for the cardinal number which applied to {△,○,□} means that it is equivalent to (one, two, three).

Thus we see the cardinal number idea as fundamental. For any set, the fundamental number question is: what is the cardinal number for the set? A basic technique for answering this question is to count the elements of the set, that is, to form, in a one after the other manner, a one-one correspondence between the set in question and the one and only set of the ordered sets (one), (one, two)... with which such a correspondence can be made, and then to give the set in question the number named by the last element in the ordered set used to count it.

ORDINAL USAGE OF COUNTING

In many situations, a particular number-object match is sought, rather than the cardinal number of some set. For example, consider a well organized, artistic socialite, who has received flowers.

"Bring me the third vase from the left from the top shelf," she directs her husband. At this point, the problem that confronts the husband is the identification of a particular vase, rather than the determination of the cardinal number of a set, even though he may count

to three in the course of the solution. This usage of number has traditionally been called the ordinal usage.

As illustrations of ordinal usage, let us consider uses of number involving a man and his automobile license, starting as he enters the License Bureau to purchase his plates. As he takes his place at the end of the line, he gains both a cardinal and an ordinal fact by a single act of counting. He learns that he is thirteenth in line, and that 13 people will be served before he returns to his car.

Finally the license plates are purchased; the number is NO 1313. Does the auto owner us 1313 cardinally or ordinally? Probably neither. To him, NO 1313 is only another temporary name for his vehicle, to be used in formal situations, such as in completing insurance forms.

A few weeks later, NO 1313's owner has a minor scrape in the parking lot, in which his bumper scrapes some paint from the fender of a parked car. Since the tiny scrape is inconsequential, he leaves the area immediately. The owner of the scraped car, however, who returns to the parking lot in time to note that NO 1313 is the license number of the departing culprit, does not feel that the scrape is inconsequential. He notifies the police, who notify the License Bureau. The Bureau uses 1313 ordinally to identify the car's owner.

Leaving NO 1313's owner to his fate, let us examine the process that the clerk uses in locating 1313 in the NO section. He finds the page containing numbers 1300-1340, puts his finger down at a guess point, which turns out to be 1308, and moves down the page to 1313.

In using this process to find 1313, the clerk uses skills based on a thorough knowledge of the counting process. When he reads his guess of 1308, he knows immediately that he must move down the page rather than

up, and that he is near enough to 1313 so that he can move his finger down the page in a quick counting-like process, rather than making a jump to a new guess.

The process of finding a particular number in a sequence of consecutive numbers occurs in the early grades as children try to find the proper page in their reading books. In gaining proficiency at this skill, children build on their ability to count. First, they learn to recognize which of any pair of numbers is larger. To see how this relates back to counting, compare it to using the dictionary. Are you sure immediately which of l and h comes earlier in the alphabet, or do you recite to yourself, "h,i,j,k,l"? Then a child must extend his knowledge of the sequence of natural numbers until, at a glance, he knows not only which of two numbers is larger, but <u>roughly</u> how far apart they are. (Note that this question, i.e. how far must I go to get from this page to the page I want? is a cardinal number question; the most direct solution for an <u>exact</u> answer would be achieved by counting the set of intervening pages.)

We see then that the counting numbers are used ordinally as well as cardinally. In what follows, sets of numbers which are not in the usual one-after-the-other sequence will be used ordinally.

ORDINAL USE OF NON-CONSECUTIVE SETS OF NATURAL NUMBERS

An auto owner's license number is, to him, usually only a name for his car. Similarly, for Dave, a casual spectator, the number on the back of a baseball player's uniform is a name, a handle, with which to keep track of a player. The symbol "☆" would do as nicely as the symbol "6". On the other hand, Fred, a less casual fan, who answers for Dave such questions as, who is #6? uses the player numbers ordinally, referring to

a list, part of which is:

4	Demeter
6	Kaline
23	Horton
24	Stanley
26	Brown
30	Northrup
31	Redmond

Even though the numbers used in the list are not consecutive, they facilitate the identification of players. But the process of locating a particular number in the sequence may be more trouble than when the numbers are consecutive. Whether one must go up or down a list such as this one to get from one number to another is decided in the same manner in which it would be decided if the list contained all the natural numbers from 1 to 31. But making even a rough estimate of how far one must go to get from one number to another is a chancey business. To use another example, consider the confusion that can occur in trying to locate an address when the house numbering jumps 1000 unexpectedly.

The use of sets of natural numbers which are in their usual order but which are not consecutive suggests that situations exist in which the "next after" or successor properties of the set of natural numbers are not essential. In fact, as the following section shows, there are circumstances in which a set (usually a set of numbers) is needed for ordering in which elements do not have successors, i.e. in which there is not an element "next after" each element.

ORDER WITHOUT SUCCESSORS

A situation somewhat similar to searching for a house by making use of its address is searching for a book in a library using its call number. As in the ordinal usage of the natural numbers, two sets are involved: the set of which we seek a particular element, i.e. the set of books in the

library, and the order keeping set, i.e. the set of call numbers. On a particular day, a call number in the set of call numbers used in the library has a successor. QA 1 .N 48 may be next after QA 1 .N 47. However, on the following day, the library may acquire a book which, according to the Library of Congress system, should be after QA 1 .N 47, but before QA 1 .N 48. This book might have been assigned the number QA 1 .N 473.

The set of natural numbers is inadequate for use as an order keeping set in situations such as this since the successor properties do not allow, for example, a number between 47 and 48. However, the natural numbers can be extended to an adequate set by introducing decimal fractions. Between 47 and 48, we can introduce 47.3. Thus, in the set of all decimal fractions, as in the set of all possible Library of Congress call numbers, it is meaningless to speak of the successor of an element.

*SIMPLE ORDER

In the preceding sections, we have seen how ordered sets of numbers are used to keep track of the elements of other sets, for example how a set of house numbers is used to keep track of a set of houses. But the idea of order itself has been analyzed only partially, by noting such facts as $(a,b,c) \neq (c,a,b)$. It is possible to be more precise about what is meant by saying that a set is ordered.

Let us use the symbol $<$ to generalize from the ideas "to the left of", "before", "less than", "earlier", "lower", etc. Let S be a set whose elements can be compared using $<$.

Definition: S is simply[1] ordered by $<$ if the following two properties hold:

(1) if x and y represent elements of S, then exactly one of these

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1. The adjective "simply" is used not to contrast with gradually ordered sets, whose definition does not involve the relation $<$, but rather, because it emphasizes that $<$ is connective.

possibilities below is true:

$$x = y \quad \text{or} \quad x < y \quad \text{or} \quad y < x \quad \text{(trichotomy property)}$$

(2) if x,y, and z represent elements of S and if

$$x < y \quad \text{and} \quad y < z$$

then it must be true that

$$x < z \quad \text{(transitivity property)}.$$

The set of natural numbers is simply ordered, but the concept of simple order, defined by the trichotomy and transitivity properties, is more general than the natural number idea. The set of all moments in time is simply ordered by the relation "before", but the elements of this set are too numerous to keep track of with the natural numbers. Another such example is the set of all points on a horizontal line with the relation "to the left of".

The formulation of simple order given by Cantor in 1895 was:

> We call an aggregate M "simply ordered" if a definite "order of precedence" (Rangordnung) rules over its elements m, so that, of every two elements m_1 and m_2, one takes the "lower" and the other the "higher" rank, and so that, if of three elements m_1, m_2, and m_3, m_1 say, is of lower rank than m_2, and m_2 is of lower rank than m_3, then m_1 is of lower rank than m_3.
>
> The relation of two elements m_1 and m_2, in which m_1 has the lower rank in the given order of precedence and m_2 the higher, is expressed by the formulae:
>
> $$m_1 < m_2 \quad , \quad m_2 > m_1.$$ [1]

*ORDINAL NUMBER

A general notion of ordinal number can be developed independently of

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1. Cantor, op. cit. p. 110

the ordinal usage of the natural numbers by using a more restricted definition of the one-one correspondence. Consider two ordered sets, A = (a,b,c) and B = (1,2,3). Ignoring order, these two sets can be shown to be equivalent by various matchings, including

However, only the second matching preserves the order, that is, only the second matching assures that if two elements in A are related by $<$, then the corresponding elements in B are also related by $<$. For example, in the second matching above, the fact that $a<b$ in A is reflected (preserved) in B by the fact that $1<2$. But in the first matching the order is not preserved, as is shown by $a<b$, but $2 \not< 1$.

Definition: Two ordered sets whose elements can be matched in such a way that order is preserved are called <u>similar</u> and are said to have the same <u>ordinal number</u>.

An example of a similarity between two simply ordered sets that children learn is the similarity between the ordered set of sounds (one, two, three,...) and the ordered set of written symbols, (1,2,3,...). Another often used similarity between two sets is the one between the set of moments in time and the points on a line:

In fact, any time an ordered set of symbols is used to denote the elements of a set (house numbers for the houses on one side of a street, etc.), a similarity between the set of symbols and the set symbolized is involved.

INDICATIONS FROM PSYCHOLOGY

Psychologists have studied the development of order ideas by children. As in the case of cardinal number ideas, they found various developmental stages. As an illustration of the type of work done, a description of part of one of the less involved experiments performed is quoted below:

> The child is given a set of ten little sticks of varying lengths and is asked to form the series from the shortest (A) to the longest (K). As in figure 11, when this has been done, he is given, one at a time and in order, nine more sticks (which

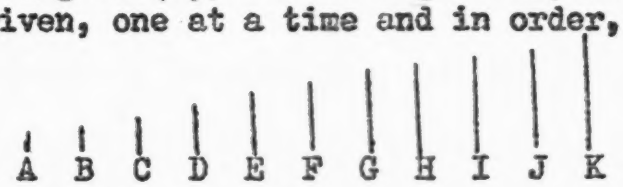

Figure 11

> we shall call a-i). He is told that these had been forgotten and are now to be inserted in their places.
> We then get the series: AaBbCcDdEeFfGgHhIiKk.[1]

The general results were:

> There are three distinct stages in the seriation of the sticks... First comes a period (both subjects whose behavior was analyzed were 4 years, 0 months) during which the child always fails to make the complete series, even for A-K, Succeeding only in making several short series which he puts side by side without regard to the order of the whole series. Or else he succeeds in building the staircase, but only considers the top of each stick, and disregards the base, and thus the total length of each element, so that his staircase is only regular at the top, and as the sticks are not placed in a horizontal line they are not in the correct order of size. In the second stage (subjects in this stage ranged from 4 years 6 months to 5 years 10 months), the child succeeds, by trial and error, in making a correct staircase, but has not acquired a system of relationships by means of which error is eliminated and extra sticks can be inserted at once in their right place. In the third stage (subjects were 6 years 0 months and 6 years 6 months), each element is without hesitation placed in a position in which it is at the same time bigger than the preceding elements and smaller than those which follow.[2]

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1. Piaget, op. cit., p. 329
2. Ibid., p. 251

However, it cannot be assumed that a child who has reached the third stage is equipped to handle all problematic situations which can be resolved into simple questions involving order. If the concreteness of the sticks of this experiment is replaced by hypothetical abstraction, the developmental level required jumps from what Piaget calls the period of "concrete operations" to that of "formal operations".

> The following neat example comes from one of Burt's tests: "Edith is fairer than Susan; Edith is darker than Lily; who is the darkest of the three?" Now this problem is rarely solved before the age of 12. Till then we find reasoning such as the following: Edith and Susan are fair, Edith and Lily are dark, therefore Lily is darkest, Susan is the fairest and Edith in between. In other words, the child of 10 reasons formally as children of 4-5 years do when serializing sticks, and it is not until the age of 12 that he can accomplish with formal problems what he could do with concrete problems of size at the age of 7, and the cause of this is simply that the premises are given as pure verbal postulates and the conclusion is to be drawn... without recourse to concrete operations.[1]

CONCLUDING REMARKS

In these units we have considered basic counting and order concepts primarily from an informal, but essentially logical point of view. Hopefully, at this point it is apparent that the ideas that young children (including kindergartners and first graders) are expected to digest are not totally obvious. Russell has said:

> The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small nor very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using "simple" in a logical sense).[2]

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1. Piaget, Jean, The Psychology of Intelligence, Routledge & Kegan Paul Ltd., London, (1950) p. 149.
2. Russell, op. cit. p.2.

Further, the difficulties in teaching basic mathematical ideas to children are only partially solved by well stated logical premises and insightful psychological experiments, although these are helpful. Stating a definition of cardinal number to a child won't help him much in learning what he needs to know about how much 13¢ is; pointing out to him that cardinal number is preserved through one-one exchanges won't convince him that 13 pennies exchanged for candies in a one-for-one manner will produce 13 candies. These things are learned through experience. A part of the art of teaching is knowing what worthwhile ideas a child is capable of learning and finding a great enough variety of experiences to allow him to gain a full grasp of those ideas.

EXERCISES AND DISCUSSION QUESTIONS

1. In the early grades, children can use counting to find how many children are present, and to see if there are enough cartons of milk for the whole class. List some other situations in which children can gain experience with counting objects.

2. The ancient Greeks used letters of their alphabet for counting. If we were to use such a scheme, what would be the dual mathematical role of the symbol "d"?

3. Why might learning the usual left to right order of written symbols conflict with learning to use rote-counting in finding the cardinal number of a set?

4. From your experience, describe some behavior showing difficulties children have in learning basic number concepts (for example, rote-counting, one-one correspondence, order, and the relation of cardinal number to counting). Suggest some activities that might help clear up these difficulties.

5. The set of children in a class can be "ordered" in many ways, for example by using their birth dates or telephone numbers. List several other ways of ordering the children in a class.

6. Sometimes a relation appears to order a set when actually it is too nebulous to do so. There is a story of a king who tried to use "more beautiful" to order his ten daughters, only to find that each was more beautiful than all the others. Children sometimes assume that such false orderings are valid; for example they assume there is an order of desirability among the available playthings. List several other false orderings.

*7. Consider the set of points pictured in Figure 12 and the relation "above".

Figure 12.

Does this relation have the trichotomy property? the transitivity property? Is it a simple order? a partial order?

*8. Give two examples of ordered sets which are partially ordered, but not simply ordered.